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L_p and Orlicz Stability of a Class of Nonlinear Timevarying Feedback Control Systems*

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INTRODUCTION

In recent years, Desoer, Sandberg, Chen, Wu *et al.* [4–6, 10, 12, 13] reported several interesting results on the question of L_p stability of a very general class of linear and a class of weakly nonlinear feedback control systems. Some partial results on L_p stability of feedback control systems with Volterra series in forward path were also reported by the author [1, 2].

In this paper, we are interested in the question of L_p and Orlicz stability of a class of strongly nonlinear-timevarying-feedback control systems having the representation

$$\left.\begin{aligned} x &= u - \lambda Ax \\ y &= Ax \end{aligned}\right\} T,$$

where u is the input, y is the output, x is the error, λ is the feedback gain, and Ax is defined by

$$(Ax)(t) = (Kfx)(t) = \int_0^t K(t, \tau) f(\tau, x(\tau))\, d\tau \tag{1.1}$$

with $t \in R_0 = [0, \infty)$. For convenience, no distinction will be made between the operators K and f and their corresponding functions.

It is assumed in this paper that (i) the Kernel $K(t, \tau)$ is a measurable function on the triangle $0 \leqslant \tau \leqslant t < \infty$ and (ii) the function $f(t, u)$ is measurable in t on R_0 for each fixed $u \in R = (-\infty, +\infty)$ and continuous in u on R for almost all $t \in R_0$ (Carathéodory condition 8, pp. 20).

It is not assumed that f satisfies a Lipschitz condition (global) in the variable u (Chen [4, pp. 192]) which is considered to be very restrictive in many practical situations.

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Sufficient conditions for L_p stability of the system T are presented through Lemma 1 and Propositions 1, 2. In Corollary 1, certain smoothness properties of the solution is investigated.

Similar results are presented for Orlicz stability of the system T in Lemma 2 and Propositions 3, 4.

2. $L_p(p \geqslant 1)$ Stability of the System T

For the proof of the L_p $(p \geqslant 1)$ stability of the system T, we need the following:

LEMMA 1. *Suppose there exist real numbers $\alpha \geqslant 1$, $p_2 \geqslant 1$ and $\beta \geqslant 0$ and a scalar valued nonnegative measurable function $g(t) \in L_{p_2}(R_0)$ so that for almost all $t \in R_0$ and all $u \in R = (-\infty, +\infty)$ $|f(t, u)| \leqslant g(t) + \beta |u|^\alpha$. Then for $p_1 = \alpha p_2$, the operator f maps $L_{p_1}(R_0)$ into $L_{p_2}(R_0)$ and is continuous and bounded.*

Proof. The first assertion of the lemma follows from the inequality $\|fx\|_{p_2} \leqslant \|g\|_{p_2} + \beta(\|x\|_{p_1})^\alpha$ for every $x \in L_{p_1}(R_0)$ which implies that $f : L_{p_1}(R_0) \to L_{p_2}(R_0)$. This in turn implies continuity (Krasnoselskii [8, Theorem 2.1, pp. 22]). Boundedness follows from the above inequality. This completes the proof of the lemma.

With the help of this lemma we can prove the following result, for which we need

DEFINITION 1. An operator A mapping a Banach space B_1 into a Banach space B_2 is said to be completely continuous on $S \subset B_1$ if it is continuous on S and AS is a compact subset of B_2 (whenever S is bounded).

PROPOSITION 1. *If the kernel $K(t, \tau)$ is measurable on the triangle $0 \leqslant \tau \leqslant t < \infty$ and satisfies the property that*

$$\hat{K}(t) \triangleq \left(\int_0^t |K(t, \tau)|^{q_2} \, d\tau\right)^{1/q_2} \in L_{p_1}(R_0)$$

(where $p_2^{-1} + q_2^{-1} = 1$) and f satisfies the hypothesis of Lemma 1, then the operator $A \triangleq Kf$ maps $L_{p_1}(R_0)$ into itself and that it is completely continuous on $L_{p_1}(R_0)$.

Proof. The first part of the proposition follows from the fact that $f: L_{p_1}(R_0) \to L_{p_2}(R_0)$ (Lemma 1) and the inequality

$$\| Ax \|_{p_1} \leqslant \| \hat{K} \|_{p_1} \| fx \|_{p_2} \tag{2.1}$$

for all $x \in L_{p_1}(R_0)$. Thus the operator A maps $L_{p_1}(R_0)$ into itself.

For the proof of complete continuity we must prove that A is continuous (on $L_{p_1}(R_0)$ and compact (i.e., maps every bounded set in $L_{p_1}(R_0)$ into a compact set in $L_{p_1}(R_0)$). The continuity of the operator A follows from that of the operator f and the inequality $\| Ax - Ay \|_{p_1} \leqslant \| \hat{K} \|_{p_1} \| fx - fy \|_{p_2}$. It remains to prove the compactness. Let $D_1 \subset L_{p_1}(R_0)$ be bounded. Since by Lemma 1, $f: L_{p_1}(R_0) \to L_{p_2}(R_0)$ is bounded $\sup_{x \in D_1} \| fx \| \leqslant d_1$ for some $0 \leqslant d_1 < \infty$. Define $D_2 = \{z = fx : x \in D_1\}$. Clearly, $D_2 \subset L_{p_2}(R_0)$ is bounded and since, for $p_2 > 1$, L_{p_2} is a reflexive Banach space, D_2 is weakly compact. If $p_2 = 1$, then we may assume that the operator f satisfies the additional property that for every $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\int_E | f(t, x(t))|^{p_2} \, dt < \epsilon \qquad \text{for all} \quad x \in D_1 ,$$

whenever the Lebesgue measure of the set $E \subset R_0$ is less than δ. Under this situation, D_2 is a weakly (sequentially) compact subset of $L_{p_2}(R_0) = L_1(R_0)$ (Dunford [7, Theorem 9, pp. 292]). Thus we may assume that D_2 is weakly sequentially compact. Now let $\{z_n = fx_n , x_n \in D_1\} \subset D_2$ be any sequence. It is clear that, for almost every $t \in R_0$,

$$\begin{aligned} y_n(t) &= \int_0^t K(t, \tau) z_n(\tau) \, d\tau \\ &\triangleq \int_0^t K_t(\tau) z_n(\tau) \, d\tau \end{aligned} \tag{2.2}$$

is defined. Further, it follows from the assumptions on K that for almost every $t \in R_0$ $K_t(\tau) \in L_{q_2}[0, t)$. Therefore, since D_2 is weakly (sequentially) compact, there is a subsequence $\{z_m\}$ $(m = n_1 , n_2 ,...)$ of $\{z_n\}$ and $z_0 \in L_{p_2}(R_0)$ so that, for almost all $t \in R_0$,

$$y_m(t) \triangleq \int_0^t K_t(\tau) z_m(\tau) \, d\tau \to \int_0^t K_t(\tau) z_0(\tau) \, d\tau \triangleq y_0(t). \tag{2.3}$$

Therefore, $\lim_{m\to\infty}(y_m(t) - y_0(t)) = 0$ almost everywhere on R_0 and, consequently,

$$\lim_{m\to\infty} |y_m(t) - y_0(t)|^{p_1} = 0, \qquad \text{a.e. on } R_0 . \tag{2.4}$$

Further,

$$|y_m(t) - y_0(t)| \leqslant \hat{K}(t)(d_1 + \| z_0 \|_{p_2}) \tag{2.5}$$

uniformly with respect to m and for almost all $t \in R_0$. By hypothesis, $\hat{K} \in L_{p_1}(R_0)$ and, therefore, $\{y_m - y_0\} \in L_{p_1}(R_0)$.

The function on the right side of the inequality (2.5) provides the dominating function required for the well-known Lebesgue dominated convergence theorem to hold. Therefore, by application of this theorem we have

$$\lim_{m\to\infty} \int_{R_0} |y_m(t) - y_0(t)|^{p_1}\, dt = \int_{R_0} \lim_{m\to\infty} |y_m(t) - y_0(t)|^{p_1}\, dt = 0. \tag{2.6}$$

Thus $y_m \to y_0$ in the strong topology of $L_{p_1}(R_0)$, and it follows from the equality $y_0 = y_0 - y_m + y_m$ and the uniform boundedness of the sequence $\{z_m\} \subset L_{p_2}$ and, consequently, that of $\{y_m\} \subset L_{p_1}$ that $y_0 \in L_{p_1}(R_0)$. Thus $AD_1 \subset L_{p_1}$ is compact, whenever $D_1 \subset L_{p_1}$ is bounded. This completes the proof. Using Lemma 1 and Proposition 1, we prove the following

PROPOSITION 2. *For an arbitrary but fixed* $r \in (0, \infty)$, *suppose the ball* $S_r = \{x \in L_{p_1}(R_0) : \| x \|_{p_1} \leqslant r\}$ *is given and let* $\sup_{x\in S_r} \| Ax \|_{p_1} = a(r)$. *Suppose the feedback gain* λ, *of the system* T *satisfying the hypotheses of the Lemma* 1 *and Proposition* 1, *is such that* $\theta(r) \triangleq r - |\lambda| a(r) > 0$.

Then, for every positive $\theta < \theta(r)$ *and for any* $u^* \in S_\theta \subset L_{p_1}(R_0)$, *the system* T *has at least one solution* $x^* \in S_r \subset L_{p_1}(R_0)$ *and that the output* $y^* \in L_{p_1}(R_0)$; *and the system* T *is locally stable in the* L_p*-sense.*

Proof. Since the operator $A : L_{p_1}(R_0) \to L_{p_1}(R_0)$ is completely continuous (Proposition 1), the number $a(r) < \infty$ for every finite $r > 0$. Consequently, for every $u^* \in S_\theta$, $\theta < \theta(r)$, the operator $B(u^*, \cdot)$ defined on $L_{p_1}(R_0)$ by $B(u^*, x) = u^* - \lambda Ax$ has the property that $\| B(u^*, x)\|_{p_1} < r$ for all $x \in S_r \subset L_{p_1}(R_0)$. Thus for every $u^* \in S_\theta$ $\theta < \theta(r)$, the operator $B(u^*, \cdot)$ maps the ball $S_r \subset L_{p_1}(R_0)$ into a subset of S_r.

Since A is a completely continuous operator (Proposition 1) acting within

$L_{p_1}(R_0)$, for every $u^* \in S_{\theta(r)} \subset L_{p_1}(R_0)$, the operator $B(u^*, \cdot)$ is also completely continuous. Thus by Schauder fixed-point principle (Krasnoselskii and Rutickii [9, pp. 209]) there is at least one solution $x^* \in S_r$ of the system T and, consequently, the output $y^* \in L_{p_1}(R_0)$, and the system T is locally stable in the L_p-sense. This completes the proof.

Suppose the linear operator K in the forward path of the system T has the representation (Bendat, [3, pp. 13])

$$(K^0 v)(t) = \int_0^t K^0(t, t-\tau) v(\tau)\, d\tau, \tag{2.7}$$

and let us denote this modified system by T'. If for almost all $\tau \in R_0$, the Kernel K^0 (of the system T') as a function of $t \geqslant \tau \geqslant 0$ and the input $u \in L_{p_1}(R_0)$ satisfy certain smoothness properties, then the solution x^* and, consequently, the output y^* of the system T' satisfy similar smoothness conditions. This is discussed in the following

COROLLARY 1. *For the system T', suppose*

$$\hat{K}^0(t) \triangleq \left(\int_0^t |K^0(t, \xi)|^{q_2}\, d\xi\right)^{1/q_2} \in L_{p_1}(R_0)$$

and is bounded uniformly on R_0 and that f satisfy the hypothesis of Lemma 1 with $1 < p_2 < \infty$. Let $S_r \subset L_{p_1}(R_0)$ be as defined in Proposition 2 and define $X_0 = \{x \in L_{p_1}(R_0) : \lim_{t\to\infty} x(t)$ exists and $\lim_{t\to\infty} |x(t)| = 0\}$. Then, if the input $u^* \in S_{\theta(r)} \cap X_0$ (where $\theta(r)$ is as defined in Proposition 2) the corresponding solution $x_0^* \in S_r \cap X_0$ and the output $y_0^* \in X_0$.

Proof. Since all the hypothesis of Lemma 1, and Propositions 1, 2 are satisfied the system T' has a solution $x_0^* \in S_r$. By hypothesis, $u^* \in S_{\theta(r)} \cap X_0$, and since $x_0^* = u^* - \lambda A x_0^* = u^* - \lambda y_0^*$, it is enough to show that $y_0^* = A x_0^* \in X_0$. Since $x_0^* \in S_r$ and, for all $t \in R_0$, $K_t^0(\xi) \triangleq K^0(t, \xi) \in L_{q_2}[0, t)$ and $\hat{K}^0(t) \in L_{p_1}(R_0)$ for every $\epsilon > 0$, there exists a $T_0 = T_0(\epsilon) \in R_0$ such that, for all $t > T_0$,

$$\begin{aligned} \int_{T_0}^t |K_t^0(\xi)|^{q_2}\, d\xi &\leqslant \epsilon^{q_2}, \\ \int_{T_0}^t |f(\xi, x_0^*(\xi))|^{p_2}\, d\xi &\leqslant \epsilon^{p_2}. \end{aligned} \tag{2.8}$$

Therefore, by Holder's inequality and the estimates (2.8) we have, for all $t > 2T_0$,

$$
\begin{aligned}
|y_0^*(t)| &= \left| \int_0^t K_t^0(t-\tau) f(\tau, x_0^*(\tau))\, d\tau \right| \\
&\leqslant \left\{ \left(\int_{t-T_0}^t |K_t^0(\xi)|^{q_2}\, d\xi \right)^{1/q_2} \left(\int_0^{T_0} |f(\xi, x_0^*(\xi))|^{p_2}\, d\xi \right)^{1/p_2} \right. \\
&\quad \left. + \left(\int_0^{t-T_0} |K_t^0(\xi)|^{q_2}\, d\xi \right)^{1/q_2} \left(\int_{T_0}^t |f(\xi, x_0^*(\xi))|^{p_2}\, d\xi \right)^{1/p_2} \right\} \\
&\leqslant \epsilon M(t), \qquad (2.9)
\end{aligned}
$$

where

$$
M(t) = \left(\int_0^{T_0} |f(\xi, x_0^*(\xi))|^{p_2}\, d\xi \right)^{1/p_2} + \left(\int_0^{t-T_0} |K_t^0(\xi)|^{q_2}\, d\xi \right)^{1/q_2}
$$

is finite for all $t > 2T_0$. Since $\epsilon > 0$ is arbitrary, $y_0^* \in X_0$ and, consequently, $x_0^* \in X_0$ also. This completes the proof of the corollary.

Remark. It appears from the above results that nonlinearities stronger than power-nonlinearity can not be handled by use of L_p spaces. This limitation, however, can be overcome to a large extent by use of appropriate Orlicz Spaces. For detailed properties of Orlicz spaces, the reader is referred to (Zaanen [14, pp. 78]), and (Krasnoselskii and Rutickii [9]).

3. Orlicz Stability of the System T

For simplicity, we shall consider the Orlicz spaces L_{M_1} and L_{M_2}, so that M_1, M_2 and the function N_2 complimentary to the function M_2 in the sense of Young (Zaanen [14, pp. 76]) satisfy the so called Δ_2 condition [9, pp. 24] i.e., there exists a finite real number $\alpha' > 0$ so that $M_i(2u) \leqslant \alpha' M_i(u)$ $i = 1, 2$, and $N_2(2u) \leqslant \alpha' N_2(u)$ for all $u \geqslant 0$).

A result analogous of that of Lemma 1 is given in the following

Lemma 2. *Suppose there exists a real number $\beta \geqslant 0$ and a real-valued measurable function $g \in L_{M_2}(R_0)$ so that*

$$
|f(t, u)| \leqslant |g(t)| + \beta M_2^{-1}[M_1(u)]
$$

and suppose the function M_2 satisfy the Δ_2 condition as defined above.

Then the operator f mpas $L_{M_1}(R_0)$ into $L_{M_2}(R_0)$ and is continuous and bounded.

Proof. The proof follows from Theorem 17.5 and Theorem 17.6 of (Krasnoselskii and Rutickii [9, pp. 174]).

Using the above lemma we obtain the following proposition analogous to Proposition 1.

PROPOSITION 3. *If the Kernel $K(t, \tau)$ is measurable on the triangle $0 \leqslant \tau \leqslant t < \infty$ and satisfies the property that*

$$\hat{K}(t) \triangleq \sup_{\rho_t(u, M_2) \leqslant 1} \left| \int_0^t K(t, \tau) u(\tau)\, d\tau \right| \in L_{M_1}(R_0)$$

(where $\rho_t(u, M_2) = \int_0^t M_2(|u(\tau)|)\, d\tau$) and f satisfies the hypothesis of Lemma 2, then the operator $A = Kf$ maps $L_{M_1}(R_0)$ into itself and that it is completely continuous on $L_{M_1}(R_0)$.

Proof. The proof follows from similar arguments as in Proposition 1 using the fact that since both M_2 and N_2 satisfy Δ_2 condition (an assumption) L_{M_2} is a reflexive Banach space (Zaanen [14, Theorem 7, pp. 158]). Therefore, a bounded subset of L_{M_2} is weakly compact. The only notable difference in the proof is that the expression (2.6) is replaced by

$$\lim_{m \to \infty} \| y_m - y_0 \|_{M_1} = \lim_{m \to \infty} \sup_{\rho(v, N_1) \leqslant 1} \left| \int_0^\infty [y_m(t) - y_0(t)] v(t)\, dt \right|, \quad (2.6)$$

where N_1 is the function complimentary to the function M_1 and

$$\rho(v, N_1) = \int_0^\infty N_1(|v(t)|)\, dt.$$

Let $v_0 \in \{v \in L_{N_1} : \rho(v, N_1) \leqslant 1\}$ be chosen so that

$$\sup_{\rho(v, N_1) \leqslant 1} \left| \int_0^\infty [y_m(t) - y_0(t)] v(t)\, dt \right| = \int_0^\infty (y_m(t) - y_0(t)) v_0(t)\, dt.$$

Clearly, $(y_m(t) - y_0(t)) \in L_1(R_0)$ and that $[y_m(t) - y_0(t)] v_0(t) \to 0$ a.e. on R_0, and $|(y_m(t) - y_0(t)) v_0(t)| \leqslant (d_1 + \| z_0 \|_{M_2}) \mid \hat{K}(t) v_0(t)|$. Since $\hat{K}(t) \in L_{M_1}(R_0)$, $\hat{K}(t) v_0(t) \in L_1(R_0)$ and we have all the conditions for Lebesgue dominated convergence theorem to hold. Thus, we have $\lim_{m \to \infty} \| y_m - y_0 \|_{M_1} = 0$. This completes the sketch of the proof.

Combining Lemma 2 and Proposition 3 and Schauder fixed-point theorem, we obtain the following result.

PROPOSITION 4. *For an arbitrary but fixed* $r \in (0, \infty)$, *suppose the ball* $S_r = \{x \in L_{M_1}(R_0) : \| x \|_{M_1} \leqslant r\}$ *is given and let* $\sup_{x \in S_r} \| Ax \|_{M_1} = a(r)$. *Suppose the feedback gain* λ, *of the system* T *satisfying the hypotheses of the Lemma* 2 *and Proposition* 3, *is such that* $\theta(r) = r - |\lambda| a(r) > 0$.

Then, for every $u^* \in S_\theta \subset L_{M_1}(R_0)$, $\theta < \theta(r)$, *the system* T *has at least one solution* $x^* \in S_r \subset L_{M_1}(R_0)$ *and that the output* $y^* = Ax^* \in L_{M_1}(R_0)$ *and the system* T *is locally Orlicz–Stable.*

For the modified system T', *a stronger result like that of Corollary* 1 *holds.*

COROLLARY 2. *For the system* T', *suppose*

$$K^0(t) \triangleq \sup_{\rho_t(v, M_2) \leqslant 1} \left| \int_0^t K^0(t, \xi) v(\xi) \, d\xi \right| \in L_{M_1}(R_0)$$

and is bounded everywhere on R_0 *and that* f *satisfies the hypothesis of Lemma* 2. *Let* $S_r \subset L_{M_1}(R_0)$ *be a closed ball of radius* r *centred at the origin, and define* $X_0 = \{x \in L_{M_1}(R_0) : \lim_{t \to \infty} x(t)$ *exists, and* $\lim_{t \to \infty} | x(t)| = 0\}$. *Then if the input* $u \in S_{\theta(r)} \cap X_0$ *(Proposition* 4), *the corresponding solution* $x_0^* \in S_r \cap X_0$ *and the output* $y_0 \in X_0$.

Remarks. It is interesting to mention that in the case of strongly nonlinear systems, as considered in this paper, Lipschitz conditions or the so-called sector conditions are not satisfied in general. If the function f appearing in the description of the system T satisfies only the conditions stated in Lemmas 1, 2, it is not possible to impose a sector condition and, consequently, Propositions 2 and 4 give sufficient conditions for only local stability in the sense of L_p or Orlicz spaces.

In fact, given the system T with f and K satisfying only the properties as stated in Lemma 1 and Proposition 1 or Lemma 2 and Proposition 3, the only choice left to the designer is the value of the feedback gain factor λ. The strength of nonlinearity even dictates the maximal class of inputs that is admissible.

Thus it is more appropriate to express the stability criterion of a system in terms of certain admissibility criterion. Precisely, corresponding to a given system T, the quadruple $\{T, V, V_i, V_0\}$ is said to be admissible if there exists a nonempty (in general) linear topological vector space V and two nonempty input and output classes V_i, $V_0 \subset V$ so that the map $T : V_i \to V_0$ is defined.

Thus, for the feedback control system T (discussed in the paper) with f and K satisfying only the conditions of Lemma 1 and Proposition 1, the quadruple $\{T, L_{p_1}(R_0), S_\theta, S_a\}$ is admissible. Similarly, if f and K satisfy only the hypotheses of Lemma 2 and Proposition 3, then the quadruple $\{T, L_{M_1}(R_0), S_\theta, S_a\}$ is admissible. The sets S_θ and $S_a \triangleq AS_r$ are contained in the appropriate space.

Conclusion

In this paper, we presented sufficient conditions for $L_p(p \geqslant 1)$ and Orlicz stability of a class of strongly nonlinear single loop feedback control systems. These results are also true for multiple loop feedback systems. The question of Orlicz stability of nonlinear feedback systems with Volterra series in the forward path is an open problem to the author.

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